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Modeling of Variable Length Towed and Tethered Cable Systems

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An algorithm is presented for modeling the dynamics of towed and tethered cable systems with fixed and varying lengths (specifically, tethers, towing, reel-in/pay-out configurations). The systems may have one or many open branches, but they must be towed or tethered from a single point. The modeling uses finite-segment (rigid-link/lumped-parameter) elements. Cable length changes (reel-in/pay-out) are modeled by having a link near the towing (or anchoring) vessel change length. The physical properties of the cable may change from link to link. The towed bodies may have control surfaces. Effects of fluid drag, lift, and buoyancy are included. Added mass forces and moments are included for the towed bodies but not for the cable itself. An illustrative application is presented for a system with three different pay-out rates.

Nomenclature

$\boldsymbol{a}_J, \boldsymbol{a}_K$ = acceleration of J, K
e_{ijk} = permutation symbol,

$$e_{ijk} = \begin{cases} 1 & i, j, k \text{ distinct and cyclic} \\ -1 & i, j, k \text{ distinct and anticyclic} \\ 0 & i, j, k \text{ not distinct} \end{cases}$$

$\boldsymbol{F}_{KM}$ = equivalent added mass force on towed body K
f_{Ki} = generalized force array defined by Eq. (10)
$\boldsymbol{I}_K$ = central inertia dyadic of towed body K
J, K = joint names
ℓ_K = length of link K
M = mean ship frame
M_{Kij} = generalized mass matrix coefficients defined by Eq. (10) and following equations
$\boldsymbol{M}_{KM}$ = equivalent added mass couple torque on towed body K
m_K = lumped mass at K
$\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ = mutually perpendicular unit vectors fixed in mean ship frame M
N_{Kij} = added mass coefficients of Eq. (17)
n = number of degrees of freedom (unconstrained system)
$\boldsymbol{n}_{K1}, \boldsymbol{n}_{K2}, \boldsymbol{n}_{K3}$ = mutually perpendicular unit vectors fixed in link or towed body K
O = origin of inertia frame R
$\boldsymbol{p}_K$ = position vector locating K relative to J
$\boldsymbol{p}_{K,\theta_{Ki}}$ = $\partial \boldsymbol{p}_K / \partial \theta_{Ki}$
Q_{Ki} = inertia frame R components of external force components on links connected at K
R = inertia reference frame
$\boldsymbol{r}_K$ = position vector locating K relative to O
$S(R, J)$ = direction cosine matrix (transformation matrix) between link J and inertia frame R
s_{Ki}, c_{Ki} = sine and cosine of θ_{Ki}
t_K = tension in link K
U_M = speed of mean ship frame
$\boldsymbol{v}_J, \boldsymbol{v}_K$ = velocity of J, K
X_{Kij} = added mass coefficients of Eq. (17)
X_M, Y_M, Z_M = mean ship frame axes
X_R, Y_R, Z_R = inertia frame axes
$\ddot{y}_{Ki}$ = lumped-mass acceleration components defined by Eq. (10)
$\boldsymbol{\alpha}_M$ = angular acceleration of mean ship from M
δ_{ij} = Kronecker delta function,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

θ_{K1}, θ_{K2} = orientation angles of link K
μ_K = reciprocal of mass m_k, $1/m_k$
ψ_M = heading angle for mean ship frame
$\boldsymbol{\omega}_M$ = angular velocity of mean ship frame M

Introduction

MODELING of submerged cable systems is an essential aspect of the design and control of towed and tethered marine systems. Such systems are typically used for surveillance and deep water measurements. They are generally long cables, possibly having several branches towed or tethered by a ship and with towed bodies at the extremities. The cables are also used to communicate with and to supply power to the towed bodies. The bodies themselves often have sensors and sonars that may be sensitive to disturbances in the steady motion of the system. Autopilots on the bodies are used to help in the control of these unwanted motions. Such complex systems require comprehensive modeling procedures to obtain meaningful simulations. Our objective here is to present and outline such a modeling procedure.

Many researchers have studied the modeling of submerged cable systems. References 1–9 provide a partial listing of such efforts. In a series of recent investigations,[4,10–12] a finite-segment multibody method was adapted to study cable systems. In these studies the cable was modeled by a series of rigid rods connected end-to-end by frictionless spherical joints. References 11 and 12 present a verification of this modeling approach through comparisons with simulations from continuum models (governed by linear partial differential equations) and by comparisons with experimental data.

With submerged cables, moving through the liquid (water) the viscous forces often far exceed both the weight forces and the inertia forces. When this happens, the distributed-mass rod model may be effectively replaced by a lumped-mass model. The lumped-mass model also consists of rigid links connected end-to-end, but the masses are concentrated at the connecting joints. In addition, the fluid and weight forces are also concentrated at the joints, thus forming a *lumped-parameter* model. In this model each rod (or cable link) is then a two-force member. Such modeling provides for numerically efficient simulation without loss of accuracy.

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In this paper we present a derivation and development of governing equations of motion for these lumped-parameter models. We generally follow the methodology of Ref. 1. The cable systems are assumed to be either towed or tethered in a marine environment. The cables themselves may be either single segment or multiple branched systems. At the cable extremities there may be towed or tethered bodies. These bodies are assumed to be rigid, but otherwise they may have distributed mass and inertia properties as well as fluid control surfaces. The cable system is assumed to be connected to a ship (or even a helicopter) whose motion is known. The cable system may have variable length to simulate reel-in/pay-out maneuvers. The length changes are taken to occur at the towed, or tethered, end of the system.

The cable system as described thus forms an *open-tree* multibody system with no closed loops. The cable extremities are connected to the towed/tethered bodies with spherical joints attached at arbitrary points of the bodies. The towed/tethered bodies may be passive, or they may be actively controlled.

In our modeling the forces on each cable link include the lift, drag, buoyancy, and weight. The same forces are included for the towed/tethered bodies as well as added mass forces generated by the bodies moving through the water.

The paper is divided into four parts with the following part developing the kinematics of the system. In the second part we examine the kinetics and thus present the governing equations of motion. The third part provides example applications, and the final part presents a discussion with concluding remarks.

Kinematics

System Configuration

Figure 1 presents a simple sketch of a towed cable system, showing coordinate axes and directions. R represents an inertial reference frame in which the system moves. M is the mean ship frame with X_M being the forward axis, Y_M the axis to the right (or starboard), and Z_M the axis downward. Initially, the mean ship axes X_M, Y_M, Z_M are assumed to be coincident with the inertia frame axes X_R, Y_R, Z_R. The mean ship frame origin thus moves in the horizontal X_R, Y_R plane with the Z_M axis remaining downward and parallel to Z_R. This movement is defined by the ship's speed U_M and heading angle ψ_M.

The position of the cable tow point and the orientations of the cable links are all defined relative to the mean ship frame M. Specifically, the tow point may have movement in all three axes directions (X_M, Y_M, Z_M) of M to account for the effects of wave-induced ship motions on the towed system.

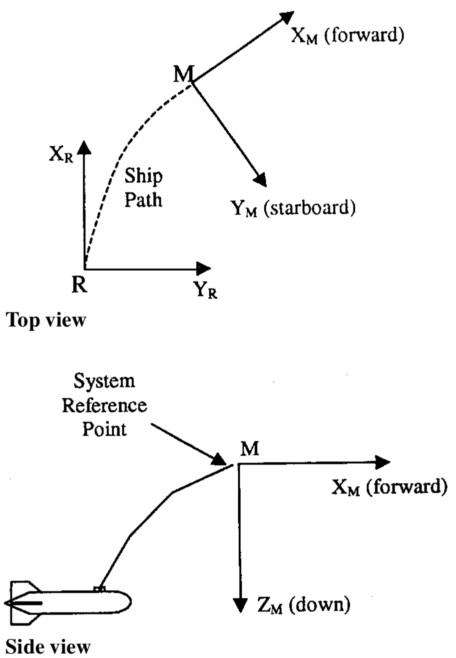

Fig. 1 Top and side views of reference frames and towed system.

The orientations of the cable links are described by body-fixed rotation sequences: Each link has a body-fixed reference frame with x, y, and z axes, with the x axis being along the link itself. When the orientation angles are all zero, these axes are aligned with those of the mean ship frame M, and the cable itself lies along the $-X_M$ axis. Then a y rotation describes motion in the X_M-Z_M plane, and a z rotation describes motion normal to this plane. Only two orientation angles are needed if cable twist is ignored.

Similarly, the towed vehicles also have body-fixed reference axes that are aligned with those of M when the orientation angles are all zero. The orientation angles themselves are given by dextral z-y-x sequences with the z rotation representing yaw, the y rotation representing pitch, and the x rotation representing roll. (Reference 13 provides a listing of the rotation matrices of the various orientation sequences.)

System Motion

Given the motion of the mean ship frame M and the motion of the tow point relative to M, as well as the orientation angles of the cable links and vehicles and the derivatives of these angles, the motion of the system will be determined.

To develop this, consider first a typical cable link K with ends J and K as in Fig. 2. Here J is taken to be at the upper end of the cable link, that is, closer to the tow point. Then the position vector $\boldsymbol{r}_K$, locating the lumped mass at K relative to the origin O of the inertia frame R, may be expressed as

$$\boldsymbol{r}_K = \boldsymbol{r}_J + \boldsymbol{p}_K \tag{1}$$

where $\boldsymbol{r}_J$ locates the mass at end J relative to O and where $\boldsymbol{p}_K$ locates end K relative to end J and may be expressed as

$$\boldsymbol{p}_K = \ell_K(-c_{K1}c_{K2}\boldsymbol{m}_1 - s_{K2}\boldsymbol{m}_2 + s_{K1}c_{K2}\boldsymbol{m}_3) \tag{2}$$

where s_{Ki} and c_{Ki} are sines and cosines of the link orientation angles θ_{K1} and θ_{K2}, measured relative to the mean ship frame M, and $\boldsymbol{m}_i$ ($i = 1, 2, 3$) are unit vectors fixed in M.

Equations (1) and (2) are used to recursively determine the positions and through differentiation, velocities, and accelerations of all of the lumped masses of the system. Similar equations may be written for the towed bodies except that here K is at the mass center of the body and J is at the cable attachment point. Then $\boldsymbol{p}_K$ is the position vector of the mass center relative to the cable attachment point.

In analyses with variable length cables (reel-in/pay-out) variable length cable links are used. Then ℓ_K is a variable, and thus $\boldsymbol{p}_K$ has a variable magnitude. (For the towed bodies $\boldsymbol{p}_K$ has a fixed magnitude.)

By differentiating in Eqs. (1) and (2), we can obtain recursive relations for the velocity and acceleration of the mass at K, i.e.,

$$\boldsymbol{v}_K = \boldsymbol{v}_J + \dot{\boldsymbol{p}}_K, \qquad \boldsymbol{a}_K = \boldsymbol{a}_J + \ddot{\boldsymbol{p}}_K \tag{3}$$

where the overdots on $\boldsymbol{p}_K$ represent derivatives in the inertia frame R. Specifically, these derivatives may be expressed in the forms

$$\dot{\boldsymbol{p}}_K = \sum_i \left(\boldsymbol{p}_{K,\theta_{Ki}}\dot{\theta}_{Ki}\right) + \boldsymbol{p}_{K,\ell}\dot{\ell}_K + (\boldsymbol{\omega}_M \times \boldsymbol{p}_K) \tag{4}$$

and

$$\ddot{\boldsymbol{p}}_K = \sum_i \left(\boldsymbol{p}_{K,\theta_{Ki}}\ddot{\theta}_{Ki} + \dot{\boldsymbol{p}}_{K,\theta_{Ki}}\dot{\theta}_{Ki}\right) + \left(\boldsymbol{p}_{K,\ell_K}\ddot{\ell}_K\right) + \left(\dot{\boldsymbol{p}}_{K,\ell_K}\dot{\ell}_K\right) + (\boldsymbol{\alpha}_M \times \boldsymbol{p}_K) + (\boldsymbol{\omega}_M \times \dot{\boldsymbol{p}}_K) \tag{5}$$

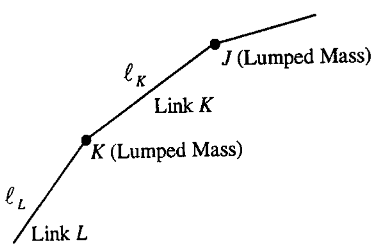

Fig. 2 Lumped mass (no branch point).

where the $\boldsymbol{p}_{K,\theta_{Ki}}$ represent the partial derivatives of $\boldsymbol{p}_K$ with respect to the link angles, where $\boldsymbol{p}_{K,\ell_K}$ is the partial derivative of $\boldsymbol{p}_K$ with respect to ℓ_K, and where $\boldsymbol{\omega}_M$ and $\boldsymbol{\alpha}_M$ are the angular velocity and angular acceleration of the mean ship frame in R. The scalar product $\boldsymbol{p}_{K,\theta_{K1}} \cdot \boldsymbol{p}_{K,\theta_2}$ is zero. From this it follows that $\ddot{\theta}_{Ki}$ may be expressed as

$$\ddot{\theta}_{Ki} = \left[\left(\boldsymbol{a}_K - \boldsymbol{a}_J - \sum_j \left(\dot{\boldsymbol{p}}_{K,\theta_{Kj}}\dot{\theta}_{Kj}\right) - \left(\boldsymbol{p}_{K,\ell_K}\ddot{\ell}_K\right) - \left(\dot{\boldsymbol{p}}_{K,\ell_K}\dot{\ell}_K\right) - (\boldsymbol{\alpha}_M \times \boldsymbol{p}_K) - (\boldsymbol{\omega}_M \times \dot{\boldsymbol{p}}_K)\right) \cdot \boldsymbol{p}_{K,\theta_{Ki}}\right] \Big/ \left(\boldsymbol{p}_{K,\theta_{Ki}} \cdot \boldsymbol{p}_{K,\theta_{Ki}}\right) \tag{6}$$

Equation (6) shows that if we know the lumped-mass accelerations, the link orientation angles and their first derivatives, the link length changes, and the ship angular motion ($\boldsymbol{\omega}_M$ and $\boldsymbol{\alpha}_M$), the second derivatives of the link orientation angles may be calculated.

For the towed bodies the derivatives of the orientation angles are related to the body-fixed components of the body angular velocity in R through the expressions

$$\dot{\theta}_{K1} = \left[\frac{(s_{K3}\omega_{K2} + c_{K3}\omega_{K3})}{c_{K2}}\right] - \dot{\psi}_M \tag{7}$$

$$\dot{\theta}_{K2} = c_{K3}\omega_{K2} - s_{K3}\omega_{K3} \tag{8}$$

$$\dot{\theta}_{K3} = \omega_{K1} + s_{K2}\frac{(s_{K3}\omega_{K2} + c_{K3}\omega_{K3})}{c_{K2}} \tag{9}$$

These equations may be differentiated to produce second derivatives of the towed-body orientation angles in terms of the orientation angles themselves and their first derivatives, and the components of the body angular velocity and angular acceleration.

Equations of Motion

The governing dynamical equations of motion are developed in a three-step process: first, each lumped mass and each towed body is isolated from the links, masses, and other bodies of the system. Newton's laws are then applied to each isolated mass and body, producing equations relating accelerations to link tensions and external forces. Next, constraint equations are written to maintain the conditions of link and towed-body rigidity. Finally, these two sets of equations are combined, forming a set of linear algebraic equations in terms of the link tensions and the cable tension components on each towed body.

Unconstrained Masses and Towed Bodies

Applying Newton's laws with each isolated mass and towed body leads to equations of the form

$$\sum_j M_{Kij}\ddot{y}_{Kj} = f_{Ki} \qquad \text{or} \qquad \ddot{y}_{Ki} = \sum_j M^{-1}_{Kij} f_{Kj} \qquad (i = 1, \ldots, n) \tag{10}$$

where n is the unrestrained number of degrees of freedom and where the $\ddot{y}_{Ki}$ represent the components of the lumped-mass acceleration (that is, the $\ddot{x}_{Ki}$) and the body-fixed components of the mass center accelerations and the angular velocity derivatives for the towed bodies (that is, the $\ddot{x}_{Ki}$ and the $\dot{\omega}_{Ki}$). The summation index j ranges from 1 to 3 for the lumped masses and from 1 to 5 for the towed bodies. The definitions of the M_{Kij} and the F_{Ki} are developed in the following paragraphs for the various components of the cable system.

Variable Length Links

Consider first the links with variable lengths. We can model cable length changes by using variable length links near the towing ship. For simplicity, it is assumed that these links are not located at cable branch points. Referring again to Fig. 2, let both links K and L have variable lengths. Then the M_{Kij} and f_{Kj} of Eqs. (10) are

$$M_{Kij} = m_K\delta_{ij} \tag{11}$$

and

$$f_{Ki} = Q_{Ki} - S(R, L)_{i1}t_L + S(R, K)_{i1}t_K \tag{12}$$

where m_K is the mass of the lumped mass, which includes half the mass of each adjoining link; the Q_{Ki} are inertia frame components of equivalent external forces acting on the adjoining links; t_J is the tension in link J; and the $S(R, J)_{ij}$ are the elements of the direction cosine matrix for link J referred to the inertia frame R. The external forces include drag, buoyancy, and weight forces[12] and are assumed to be constant over each link. Note that these equations are the same as for fixed length links except that the mass m_K changes as the link lengths change. Here it is assumed that the thrust effect caused by the changing mass is negligible.

Branch Point Lumped Masses

Consider next lumped masses at branching points, as depicted in Fig. 3. For simplicity, all adjoining links to the branch point are assumed to have fixed lengths. In this case the M_{Kij} and the f_{Kj} of Eqs. (10) are

$$M_{Kij} = m_K\delta_{ij} \tag{13}$$

and

$$f_{Ki} = Q_{Ki} + S(R, K)_{i1}t_K - \sum_\ell S(R, L_\ell)_{i1}t_{L_\ell} \tag{14}$$

where the nomenclature is the same as in Eqs. (11) and (12).

Lumped Mass at a Towed Body

Consider next a lumped mass K attached to a towed body as depicted in Fig. 4. The assumption is made that no branch points occur at the towed body and that the adjoining cable link has a fixed length. In this case the M_{Kij} and f_{Ki} of Eqs. (10) are

$$M_{Kij} = m_K\delta_{ij} \tag{15}$$

and

$$f_{Ki} = Q_{Ki} + S(R, K)_{i1}t_K + \sum_j S(R, L)_{ij}t_{L_j} \tag{16}$$

where here m_K is half the mass of the adjoining link, the Q_{Ki} are inertia frame components of half the resultant external force acting on link K, and the t_{Lj} are towed-body frame components of the force the towed body exerts on the cable. The remaining nomenclature is the same as in Eqs. (11) and (12).

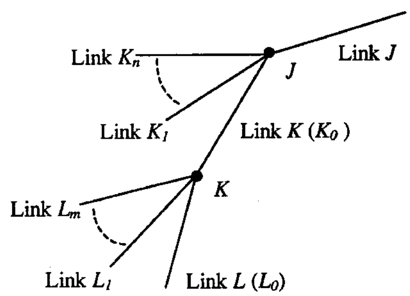

Fig. 3 General lumped mass with branch points.

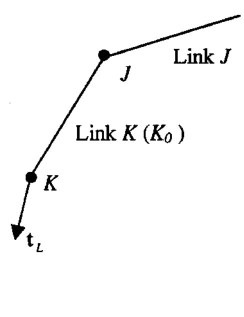

Fig. 4 Lumped mass K attached to a towed vehicle L.

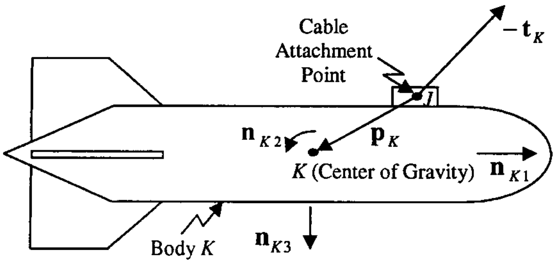

Fig. 5 Towed vehicle with cable attachment point.

Towed Body

Finally, consider a towed body as depicted in Fig. 5. Let the unit vectors $\boldsymbol{n}_{Ki}$ ($i = 1, 2, 3$), fixed in the body, represent forward, starboard, and downward directions as shown in Fig. 5. Let $\boldsymbol{t}_K$ (expressed in component form as $\sum_i t_{Ki}\boldsymbol{n}_{Ki}$) represent the equivalent force the towed body exerts on the attached cable link. Here K is at the mass center of the towed body. Let $\boldsymbol{v}_{K/F}$ (expressed in component form as $\sum_i u_{Ki}\boldsymbol{n}_{Ki}$) and $\boldsymbol{\omega}_K$ (expressed in component form as $\sum_i \omega_{Ki}\boldsymbol{n}_{Ki}$) represent the velocity of the mass center of towed body K relative to the fluid and the angular velocity of towed body K, respectively. Then the equivalent fluid drag and lift force and moments are assumed to be functions of the components u_{Ki} and ω_{Ki} and hydrodynamic coefficients characterizing the towed body. Let the equivalent added mass force and moments $\boldsymbol{F}_{KM}$ and $\boldsymbol{M}_{KM}$ be expressed in the forms

$$\boldsymbol{F}_{KM} = \sum_{i,j} X_{Kij}\dot{u}_{Kj}\boldsymbol{n}_{Ki}, \qquad \boldsymbol{M}_{KM} = \sum_{i,j} N_{Kij}\dot{\omega}_{Kj}\boldsymbol{n}_{Ki} \tag{17}$$

where the X_{Kij} and the N_{Kij} are added mass coefficients. Using this notation, the M_{Kij} and the f_{Kj} of Eqs. (10) are

$$M_{Kij} = \begin{cases} m_K\delta_{ij} - X_{Kij} & i = 1, 2, 3;\ j = 1, 2, 3 \\ -X_{Kij} & i = 1, 2, 3;\ j = 4, 5, 6 \\ -N_{Kij} & i = 4, 5, 6;\ j = 1, 2, 3 \\ I_{K(i-3)(j-3)} - N_{K(i-3)j} & i = 4, 5, 6;\ j = 4, 5, 6 \end{cases} \tag{18}$$

and

$$f_{Ki} = \begin{cases} Q_{Ki} - t_{Ki} + \sum_j X_{Kij}(\boldsymbol{\omega}_K \times \boldsymbol{v}_{K/F}) \cdot \boldsymbol{n}_{Kj} & i = 1, 2, 3 \\ M_{KT(i-3)} + M_{KQ(i-3)} & \\ \quad -(\boldsymbol{\omega}_K \times \boldsymbol{I}_K \cdot \boldsymbol{\omega}_K) \cdot \boldsymbol{n}_{K(i-3)} & \\ \quad + \sum_j N_{K(i-3)j}(\boldsymbol{\omega}_K \times \boldsymbol{v}_{K/F}) \cdot \boldsymbol{n}_{Kj} & i = 4, 5, 6 \end{cases} \tag{19}$$

where the M_{KT} are the components of the moment of $-\boldsymbol{t}_K$ about mass center K, the M_{KQj} are components of the moment of the external forces about K, and the I_{Kij} are components of the central inertia dyadic.[14] All components in Eqs. (18) and (19) are referred to the body-fixed unit vectors $\boldsymbol{n}_{Ki}$ ($i = 1, 2, 3$).

Constraint Equations

The foregoing dynamics equations need to be augmented with constraint equations to maintain the connectivity of the system. Although the vast majority of the cable links have fixed length, those near the towing ship may have variable lengths to model deployment and retrieval. Referring to Fig. 2, we see that in either event the constraint equation maintaining the connection between the lumped masses J and K is

$$\boldsymbol{p}_K \cdot \boldsymbol{p}_K = \ell_k^2 \tag{20}$$

where $\boldsymbol{p}_K$ locates K relative to J. By differentiating, we can express the constraint in terms of accelerations as

$$(\boldsymbol{a}_J - \boldsymbol{a}_K) \cdot \boldsymbol{n}_{K1} = \ell_K^2\dot{\boldsymbol{n}}_{K1}^2 - \ddot{\ell}_K \tag{21}$$

where the unit vector $\boldsymbol{n}_{K1}$ is directed from K to J.

The towed bodies are connected to the cable system by spherical joints. Referring to Fig. 5, we see that the rigidity of the towed body is maintained by the expression

$$\boldsymbol{a}_K = \boldsymbol{a}_J + \boldsymbol{\alpha}_K \times \boldsymbol{p}_K + \boldsymbol{\omega}_K \times (\boldsymbol{\omega}_K \times \boldsymbol{p}_K) \tag{22}$$

which in component form may be expressed as

$$\sum_j S(R, K)_{ji}\ddot{x}_{Jj} - \ddot{x}_{Ki} + \sum_{j,k} e_{jki}p_{Kk}\dot{\omega}_{Kj} = -[\boldsymbol{\omega}_K \times (\boldsymbol{\omega}_K \times \boldsymbol{p}_K)] \cdot \boldsymbol{n}_{Ki} \tag{23}$$

where the $\ddot{x}_{Jj}$ are inertia frame components of the acceleration of lumped mass J, the $\ddot{x}_{Ki}$ and the $\dot{\omega}_{Ji}$ are the $\boldsymbol{n}_{Ki}$ components of the acceleration of the mass center and the angular acceleration of the vehicle, the p_{Ki} are the $\boldsymbol{n}_{Ki}$ components of $\boldsymbol{p}_K$, and e_{ijk} is the standard permutation symbol.

Cable Tensions

To calculate the cable tensions, we may solve the dynamics equations for the inertia frame components of the lumped-mass accelerations, the $\boldsymbol{n}_{Ki}$ components of the towed-body mass center accelerations, and the $\boldsymbol{n}_{Ki}$ components of the towed-body angular accelerations. These results may then be substituted into the constraint equations resulting in a set of equations that is linear in the cable tensions and in the cable tension components on the towed bodies. There is one tension for each cable link and three tension components for each towed body.

The coefficient matrix of the link tensions and the towed-body tension components is sparse with most nonzero elements clustered along the diagonal. The matrix is tridiagonal for systems with no branch points and no towed bodies. Towed bodies increase the bandwidth by creating a 4×4 block on the diagonal. Branch points give rise to elements well off the diagonal. Because most systems have only a few branch points, however, these off-diagonal elements can be isolated and treated individually, and thus they do not adversely affect the solution process.

A tension equation for a variable length link (whose adjoining links are also of variable length) may be obtained by substituting from Eqs. (10) and (11) into Eq. (21), leading to

$$-\mu_J(\boldsymbol{n}_{K1} \cdot \boldsymbol{n}_{J1})t_J + (\mu_J + \mu_K)t_K - \mu_K(\boldsymbol{n}_{K1} \cdot \boldsymbol{n}_{L1})t_L = \ell_K\dot{\boldsymbol{n}}_{K1}^2 - \ddot{\ell}_K + \sum_j \big[\mu_J S(R, K)_{j1}Q_{Jj} - \mu_K S(R, K)_{j1}Q_{Kj}\big] \tag{24}$$

where μ_K is $1/m_K$, the unit vector $\boldsymbol{n}_{K1}$ is directed from K to J, and the unit vector $\boldsymbol{n}_{L1}$ is directed from L to K. In this expression we have inverted the diagonal mass matrix of Eq. (11).

Equation (24) may be applied with any cable link where no branches occur. If a link K has fixed length, then the derivatives $\dot{\ell}_K$ are zero.

Cable lengths that are adjacent to branch points or to towed bodies are assumed to have fixed length. Then a tension equation for links with branch points (see Fig. 3) may be obtained by substituting from Eqs. (10), (13), and (14) into Eq. (21) with $\ddot{\ell}_K$ being zero, leading to the expression

$$-\mu_J(\boldsymbol{n}_{K1} \cdot \boldsymbol{n}_{J1})t_J + \mu_J \sum_m \big(\boldsymbol{n}_{K1} \cdot \boldsymbol{n}_{K_m1}\big)t_{K_m} + (\mu_J + \mu_K)t_K - \mu_K \sum_m \big(\boldsymbol{n}_{K1} \cdot \boldsymbol{n}_{L_m1}\big)t_{L_m} = \ell_K\dot{\boldsymbol{n}}_{K_1}^2 + \sum_j \big[\mu_J S(R, K)_{j1}Q_{Jj} - \mu_K S(R, K)_{j1}Q_{Kj}\big] \tag{25}$$

In like manner a tension equation for a link attached to a towed body (see Fig. 4) may be obtained by substituting Eqs. (10), (15), and (16) into Eq. (21) with $\ddot{\ell}_K$ being zero, leading to

$$-\mu_J(\boldsymbol{n}_{K1}\cdot\boldsymbol{n}_{J1})t_J+(\mu_J+\mu_K)t_K-\mu_K\sum_k(\boldsymbol{n}_{K1}\cdot\boldsymbol{n}_{Lk})t_{Lk}$$
$$=\ell_K\dot{\boldsymbol{n}}_{K1}^2+\sum_j\left[\mu_J S(R,K)_{j1}Q_{Jj}-\mu_K S(R,K)_{j1}Q_{Kj}\right] \tag{26}$$

Finally, a tension equation for a towed body may be obtained by substituting Eqs. (10), (18), and (19) into Eq. (23). After expansion and simplification this leads to

$$\mu_J(\boldsymbol{n}_{K1}\cdot\boldsymbol{n}_{J1})t_J+\sum_m\tilde{M}_{Kim}t_{Km}=[\boldsymbol{\omega}_K\times(\boldsymbol{\omega}_K\times\boldsymbol{p}_K)]\cdot\boldsymbol{n}_{Ki}$$
$$-\mu_J\sum_j S(R,K)_{ji}Q_{Jj}+\sum_k\left\{M^{-1}_{Kik}-\sum_j\left[P_{Kji}M^{-1}_{K(j+3)k}\right]\right\}$$
$$\times\left\{Q_{Kk}+\sum_\ell[X_{Kk\ell}(\boldsymbol{\omega}_K\times\boldsymbol{v}_{K/F})\cdot\boldsymbol{n}_{K\ell}]\right\}+\sum_k\left\{M^{-1}_{Ki(k+3)}\right.$$
$$\left.-\sum_j\left[P_{Kji}M^{-1}_{K(j+3)(k+3)}\right]\right\}[\boldsymbol{M}_{KQ}-(\boldsymbol{\omega}_K\times\boldsymbol{I}_K\cdot\boldsymbol{\omega}_K)]\cdot\boldsymbol{n}_{Kk}$$
$$+\sum_k\left\{M^{-1}_{Ki(k+3)}-\sum_j\left[P_{Kji}M^{-1}_{K(j+3)(k+3)}\right]\right\}$$
$$\times\left\{\sum_\ell[N_{Kk\ell}(\boldsymbol{\omega}_K\times\boldsymbol{v}_{K/F})\cdot\boldsymbol{n}_{K\ell}]\right\} \tag{27}$$

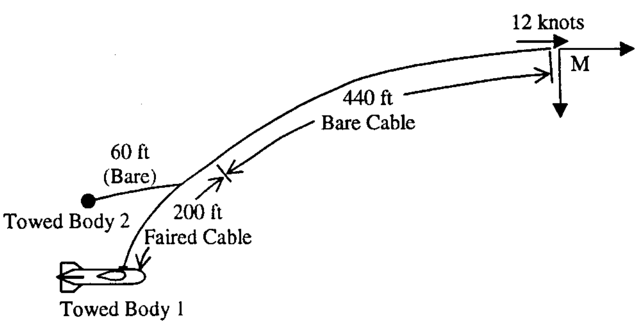

Fig. 6 Example towed system.

where $\tilde{M}_{Kim}$ and P_{Kji} are

$$\tilde{M}_{Kim}=\mu_J\delta_{im}+M^{-1}_{Kim}-\sum_j M^{-1}_{Ki(j+3)}P_{Kjm}$$
$$-\sum_j P_{Kji}M^{-1}_{K(j+3)m}+\sum_{j,k}P_{Kji}M^{-1}_{K(j+3)(k+3)}P_{Kkm} \tag{28}$$

and

$$P_{Kji}=\sum_k e_{jki}p_{Kk} \tag{29}$$

Equation (27) is to be evaluated for $i=1,2,3$ and thus represents three equations. (Observe that the mass matrix M_K is a constant 6×6 matrix and needs only to be inverted once. Observe further that $\tilde{M}_k$ is a constant 3×3 matrix and needs only to be calculated once.)

Numerical Solution Procedure

Given 1) the physical and geometrical data of the system, 2) the motion of the mean ship frame, 3) the motion of the cable tow point relative to the mean ship frame, 4) the cable length changes, and 5) the initial configuration and the initial movement of the system, the following procedure may be used to determine the movement of the system at a sequence of subsequent time steps:

1) Compute the position vectors and the velocity vectors of the cable lumped masses and the mass centers of the towed bodies. Also compute the angular velocities of the towed bodies.

2) Evaluate the external forces acting on the system.

3) Develop the set of linear algebraic equations for the tensions using Eqs. (24–29). Solve these equations using a Gaussian elimination procedure, which can exploit the sparseness of the coefficient matrix.

4) Compute the acceleration of the cable lumped masses and the angular accelerations of the towed bodies using Eqs. (10–19).

5) Compute the second derivatives of the orientation angles of the cable links using Eq. (6). Also, compute the second derivatives of the towed-body orientation angles using Eqs. (7–9). [Note that Eqs. (7–9) must be differentiated to obtain the second derivatives.]

6) Numerically integrate the expressions of Step 5 to obtain values of the orientation angles and their derivatives at an incremental time step.

7) Using these results, repeat Steps 1 to 6 for all subsequent time increments.

The assumption is made here that the number of finite segments is held constant throughout a simulation. To reel the cable in, the link at the towed (or tethered) end is shortened. As this link reaches its minimum length, the next cable link is shortened. To pay cable out, at least one short link is located at the towed end of the cable

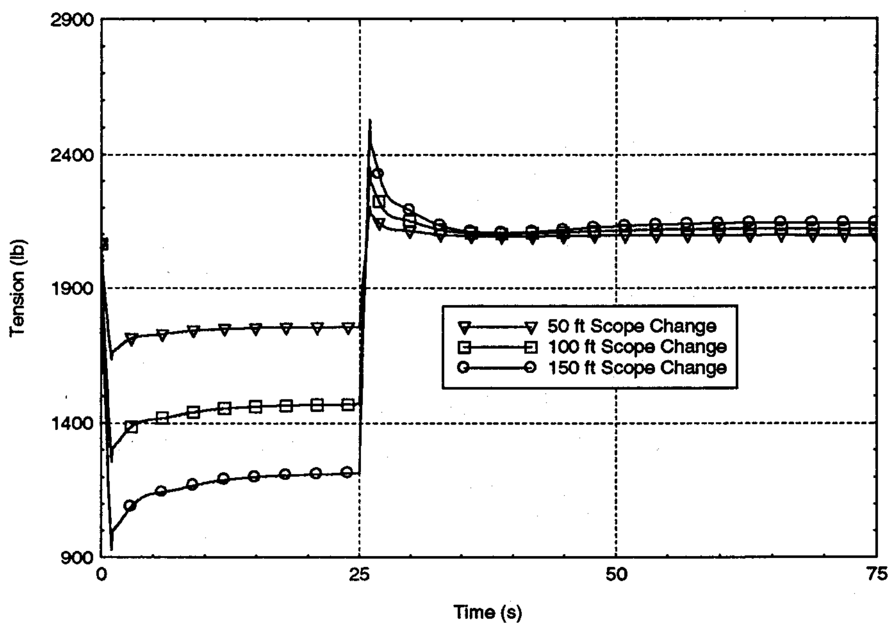

Fig. 7 Tension at the ship for various scope changes.

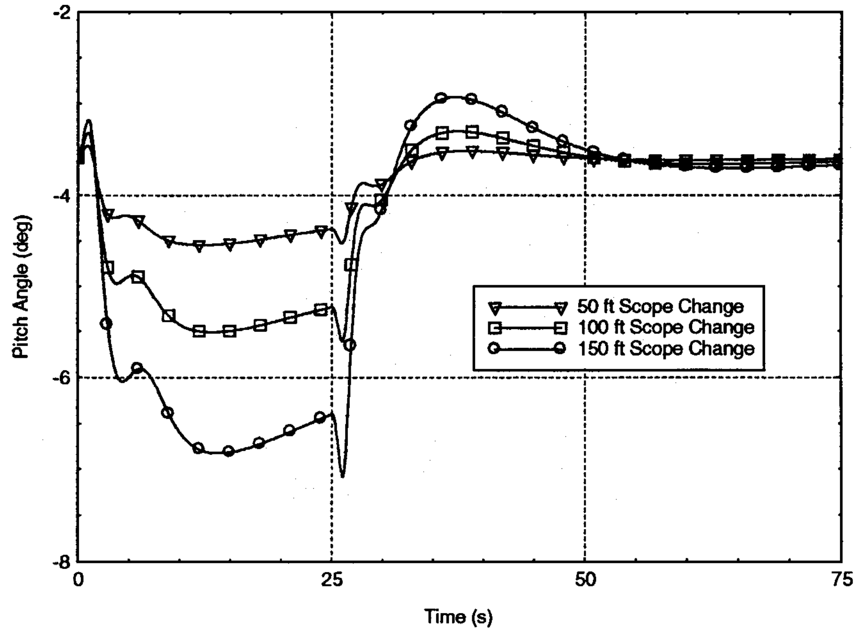

Fig. 8 Pitch angle of towed body 1 for various scope changes.

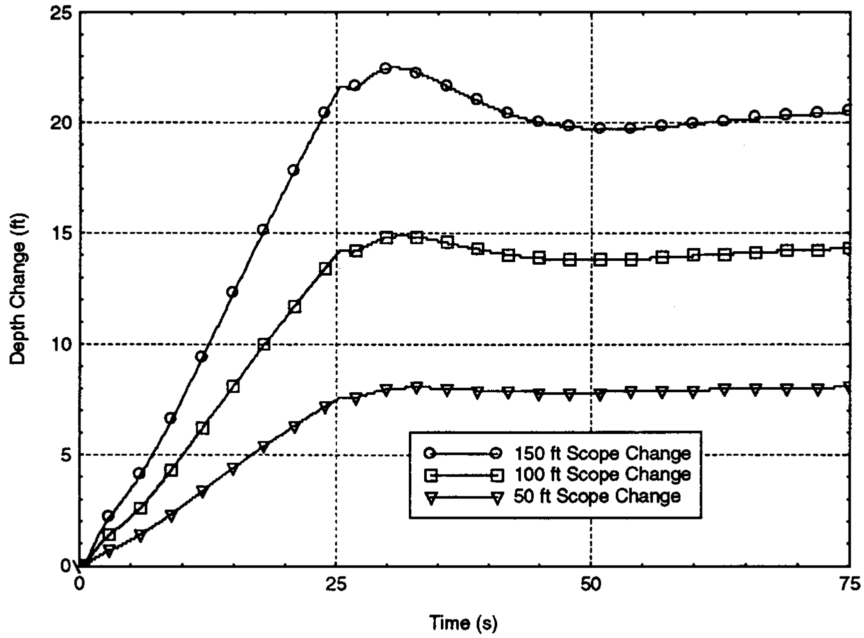

Fig. 9 Depth changes of towed body 1 for various scope changes.

model. If a large amount of cable is to be deployed, then some short links may be included. In this case the short link located farthest from the towed end is expanded first until it reaches its maximum length. Then the next closest link is expanded. The short link at the towed end is expanded last in this sequence. In this way pay-out and reel-in occur very near the towed end of the cable. In each case minimum and maximum link lengths are specified to maintain the integrity of the model.

Numerical Simulations

The foregoing procedures have been developed into algorithms for a Fortran computer program called DYNTOCABS (Dynamics of Towed Cable Systems). The following paragraphs describe an example illustrating some of the capabilities of DYNTOCABS. Although the focus of the example is upon open-loop nonlinear dynamic analysis, DYNTOCABS is also capable of steady-state, open-loop linear dynamic analysis as well as closed-loop nonlinear dynamic analyses. Closed-loop analysis is particularly useful for testing towed-body autopilot designs.

Figure 6 presents a sketch of the example system. It has two branches and two towed bodies. The main branch has a variable length with the initial length being 640 ft (195 m). To illustrate the capability of accommodating variable physical properties, the last 200 ft (67 m) have drag-reducing fairing. The secondary branch has a fixed length of 60 ft (18.3 m). The towed body at the end of the main branch has a wing producing a depressive force, and the cable attachment point is above and in front of the mass center. The towed body at the end of the secondary branch has no wings, and the cable attachment is assumed to be at the mass center.

In the simulation the system is initially towed at 12 knots (20.26 ft/s or 6.175 m/s). The scope of the system is then increased by a pay-out at the ship tow point. This pay-out is modeled by a variable length cable link at the ship tow point. The pay-out rates are increased linearly for a 1-s interval reaching pay-out speeds of 2, 4, and 6 ft/s (0.6, 1.2, and 1.8 m/s) in three different simulations. After 1 s the pay-out rates are held constant and then decreased linearly to zero again during a 1-s time interval. The three pay-out rates produce overall cable length increases of 50, 100, and 150 ft

(15.24, 30.48, and 45.7 m), respectively. Figures 7–9 show the cable tension at the ship and the pitch and depth of the main branch towed body.

Observe that the tensions drop dramatically during pay-out. The tension is a maximum when pay-out stops but then assumes a somewhat reduced steady-state value. During the pay-out period, the main branch towed body pitches up and then down. When the pay-out is concluded, the body transitions to a steady-state pitch angle. Finally, the body steadily increases depth with pay-out, but assumes a steady-state depth once pay-out is concluded. The depth changes for the secondary branch towed body (not shown) follow closely those of the main branch towed body. The faster pay-out rates generally produce larger overshoots with longer settling times.

Discussion and Conclusions

Taken as a whole, a variable length, multiple branch, towed cable system with towed bodies and variable hydrodynamic effects is clearly a complex dynamical system. Meaningful quantitative analyses of such systems are intractable without resorting to finite-segment modeling. Accordingly, the foregoing procedures are intended to provide such modeling with the objective of accurate and efficient simulation of the cable system dynamics.

As with finite element modeling, the cable is represented as a chain of finite-length segments, which in turn are represented by lumped masses. This modeling is believed to be both accurate and efficient while at the same time comprehensive and applicable to a broad range of cable systems and configurations.

A summary description of the modeling and analysis capabilities of the presented procedures is listed in the following text.

A model description consists of the following:

1) Finite-length links from chains representing the cable system are present.

2) The links are connected with frictionless spherical joints.

3) Link masses are lumped at the link joints.

4) The cable link system is towed or anchored in liquid (water)/gas (air) media by having an end attached to a tow point.

5) The tow point is positioned in a horizontally moving reference frame representing the towing ship.

6) The tow point can have oscillatory motion in the ship frame to simulate the effects of water waves or air turbulence.

7) Variable length links are used near the tow point to represent cable deployment and retrieval.

8) Towed bodies are modeled as submersible vessels with arbitrary hydrodynamic properties.

9) Multiple branching is used to simulate systems with several towed bodies.

The analysis capabilities are listed next:

1) Cables can have variable or fixed length to simulate towing or anchoring.

2) Cables can have multiple branches.

3) Cables can have variable hydrodynamic properties along the length to represent gas/liquid, lift, drag, buoyancy, and added mass.

4) Linear and nonlinear transient and steady-state analyses can be conducted.

The example presented is intended to illustrate the kind of analyses that can be obtained using the presented procedures. The numerical results show that pay-out allows the towed systems to reach increased depth and that sudden arresting of the pay-out creates cable tension impulses and increased time to reach dynamic equilibrium.

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